The Vickrey-Clarke-Groves Mechanism

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Auctions are often about selling multiple items at the same time: timber, spectrum, oil fields, etc.

But it is often the case that the buyers want only a **combination** of the items (a subset of the items). However,

- bidders may want to participate in more auctions then the number of items they want;
- bidders do not want to buy more items than needed.

Remark: Bidders are not obliged to buy several items: 1 item only is also a combination.

The **Vickrey-Clarke-Groves (VCG)** auction is the solution to this problem.

- $B = \{1, 2, \ldots, n\}$ is the set of bidders
- $X = \{x_1, x_2, \ldots, x_k\}$ is the set of items to be sold.

Each bidder i has a valuation v_i over the **sets** of items.

Suppose that bidder 1's most preferred is to have items x_1 and x_2, and then x_2 and x_3, and then x_1, x_2, x_3. His/her valuation would be then (for instance)

- $v_1(\{x_1, x_2\} = 20$
- $v_1(\{x_2, x_3\} = 15$
- $v_1(\{x_1, x_2, x_3\} = 10$

More generally, each bidder has a **valuation function** that assigns to each **set of items** a value.

So bidder i's valuation is a function v_i that assigns for each subset $S \subseteq X$ a valuation $v_i(S)$.

The interpretation is the same as before: $v_i(S)$ is how much bidder i is willing to pay to get the items S:

- all the items that are contained in S
- no item that is not contained in S.

The VCG auction works as follows:

1. Each bidder submits his bid function.
   $\Rightarrow$ it is like a sealed-bid auction.
2. The assignment (who gets what) is the one that maximizes the **social value**.
3. Each bidder pays the **externality** she imposes on the other bidders.

Maximizing the social value

Suppose we are the auctioneer and we already collected each bidder's bid functions.

The first objective is to find an **assignment** of the items (i.e., which item goes to whom) $\mu : N \to 2^S$:

$$\mu(i) = \text{the set of items that goes to bidder } i.$$

The assignment μ must be such that **no two bidders get assigned the same item** :

If $x \in \mu(i)$ (i.e., bidder i gets item x), then there is no bidder $j \neq i$ such that $x \in \mu(j)$.

The assignment of interest is the assignment μ^* that maximizes the **social value**:

$$\sum_{i \in N} v_i(\mu^*(i)) \;\geq\; \sum_{i \in N} v_i(\mu(i)) \qquad \text{for any other assignment } \mu$$

where for each bidder i, v_i is the bid function submitted by i.

- $\mu^*(i)$ is the assignment of bidder $i \rightarrow$ a set of items
- $v_i(\mu^*(i))$ is how much bidder i claimed she values the set of items $\mu^*(i)$.

Why maximizing the social value?

Take any auction,

- v is the value of the buyer
- p is the price

$$\text{social surplus} = \underbrace{p}_{\text{seller's surplus}} + \underbrace{(v-p)}_{\text{winner's surplus}} + \underbrace{(n-1)\times 0}_{\text{losers' surplus}}$$
$$= v$$

For standard auctions (2nd price, Dutch, etc.) and environments (private and independent values):

- Winner = bidder with the highest valuation

  $\Rightarrow$ social surplus maximized.

VCG follows the same principle: to implement the assignment that maximizes social value/surplus.

Paying the externality

Once we have the optimal assignment we do for each bidder i the following:

- Consider the set of all bidders except bidder i: $N \backslash \{i\}$
- Find the assignment μ of the items in S that maximizes the social value over the set $N \backslash \{i\}$:

$$\max \sum_{j \neq i} v_j(\mu(j))$$

The assignment that maximizes this sum is the one that would prevail if bidder i were not present.

The price that bidder i has then to pay for the items in $\mu^*(i)$ is then

$$\underbrace{\max \sum_{j \neq i} v_j(\mu(j))}_{\substack{\text{others' social value} \\ \underline{\text{without}} \text{ bidder } i}} \quad - \quad \underbrace{\sum_{j \neq i} v_j(\mu^*(j))}_{\substack{\text{others' social value} \\ \underline{\text{with}} \text{ bidder } i}}$$

Example

Suppose there are two bidders, 1 and 2, and two items, x_1 and x_2. Assume that the valuations are

	Bidder 1	Bidder 2
x_1	10	5
x_2	5	3
x_1 & x_2	11	6

For each possible assignment we compute the social value:

	Bidder 1	Bidder 2	social value
assignment ♯1	x_1, x_2		11
assignment ♯2		x_1, x_2	6
assignment ♯3	x_1	x_2	13
assignment ♯4	x_2	x_1	10

The assignment that maximizes the social value is then assignment ♯3, so we have

$$\mu^*(1) = x_1 \qquad \mu^*(2) = x_2 \ .$$

- If bidder 1 is absent, bidder 2 can get both items and her valuation is 6.
- When bidder 1 is present bidder 2 gets only x_2, with a valuation of 3.
- So bidder 1's price for x_1 is $6 - 3 = 3$

The assignment that maximizes the social value is then assignment $\sharp 3$, so we have

$$\mu^*(1) = x_1 \qquad \mu^*(2) = x_2 \ .$$

- If bidder 2 is absent, bidder 1 can get both items and her valuation is 11.
- When bidder 2 is present bidder 1 gets only x_1, with a valuation of 10.
- So bidder 2's price for x_2 is $11 - 10 = 1$

When there is only one item to be sold, the VCG auction is simply the Vickrey auction:

- $i =$ bidder with the highest valuation
- $j =$ bidder with the second highest valuation.

- If i is here: assignment that maximizes the social value:
  - i **gets the item.**
  - j gets nothing.
  - All other bidders get nothing.

- If i is NOT here: assignment that maximizes the social value:
  - i does **not** get the item (she's not here).
  - j **gets the item**.
  - All other bidders get nothing.

Let v_j = valuation of bidder j.

$= (\mathsf{v}_j + 0 + \cdots + 0) - (0 + 0 + \cdots + 0)$
$= \mathsf{v}_j$ = 2nd highest valuation

$\Rightarrow$ The winner pays the second highest valuation.

Remark

*The VCG **auction** presented here is a special case of a more general object, the VCG **mechanism**. To make the distinction some authors called this VCG auction the **generalized Vickrey auction**.*

Truthful bidding under VCG

The auctioneer does not know the bidders' valuations. She only has the bids: everything is calculated using the bids.

Bidders may not be forced to submit their true valuation functions.

It turns out that submitting one's **true** valuation function is a **dominant strategy** in the VCG auction.

Proof

Suppose that bidder i bids $\widehat{v}_i(\cdot)$ instead of her true valuation, $v_i(\cdot)$. Define:

- $\widehat{\mu}$ is the assignment that maximizes

$$(v_1, v_2, \ldots, v_{i-1}, \widehat{v}_i, v_{i+1}, \ldots, v_n)$$

- $\widehat{V}_{N\setminus\{i\}} = \sum_{j\neq i} v_j(\widehat{\mu}(j))$
- $V_{N\setminus\{i\}} = \max_\mu \sum_{j\neq i} v_j(\mu(j)) \leftarrow$ **Does not depend on i's bid**
- $V^*_{N\setminus\{i\}} = \sum_{j\neq i} v_j(\mu^*(j))$

Let

$$\begin{aligned} U_i &= \text{net payoff when bidding truthfull, } v_i(\cdot) \\ &= v_i(\mu^*(i)) - (V_{N\setminus\{i\}} - V^*_{N\setminus\{i\}}) \end{aligned}$$

and

$$\begin{aligned} \widehat{U}_i &= \text{net payoff when bidding } \widehat{v}_i(\cdot) \\ &= v_i(\widehat{\mu}(i)) - (V_{N\setminus\{i\}} - \widehat{V}_{N\setminus\{i\}}) \end{aligned}$$

Compute the difference

= <4->$(\mathsf{v}_i(\mu^*(i)) + V^*_{N\setminus\{i\}})\sum_j v_j(\mu^*(j)) - < 4 - >$

$$(v_i(\widehat{\mu}(i)) + \hat{V}_{N\setminus\{i\}})\sum_j v_j(\widehat{\mu}(j))$$

Recall that μ^* is the assignment that maximizes the social value. So,

$$\sum_j v_j(\mu^*(j)) - \sum_j v_j(\widehat{\mu}(j)) \quad \Leftrightarrow \quad U_i - \widehat{U}_i \geq 0$$

That is, bidder i's highest net payoff is higher when bidding truthfully.

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Problem with the VCG auction

In practice, the VCG auction is not used in many cases. Why?

- A crucial step is to find an assignment that maximizes the social welfare
- Amounts to compare all possible assignments: this can be a **gigantic** number
- We also need to ask bidders to submit a valuation function over a **large** number of possibilities.

  If there are 10 objects, the bidder would have to submit $2^{10} = 1024$ different valuations.

Example

- 10 bidders, for 10 items: each bidder only want 1 item.
- To find the optimal assignment μ^*:
  $10! = 3,628,800$ combinations
- For each bidder, the optimal assignment when not present:
  $10! = 3,628,800$ combinations
- So to compute payoffs one need:
  $10! + 10 \times 10! = 39,916,800$

Computer scientists say that this problem is "NP complete":

- we don't know how to find a solution in "reasonable time"
- If we multiply by 2 the size of the problem (number of bidders and objects) the number of calculations increases by a factor more than 2 (exponential).

Take-away

- **Combinatorial auctions** address the problem of selling multiple items at the same time.
- Bidder's valuations in such settings specify a value **for each** combination of items.
- In the Vickrey-Clarke-Groves auction we first assign the items so that the **social valuation** is the highest (the **optimal valuation**).
- Each bidder pays the **externality** she imposes on the others:

$$\begin{aligned}\text{price} = &\text{ others' max social value without the bidder} \\ &- \text{others' social value at optimal assignment}\end{aligned}$$

- The VCG auction **generalizes** the Vickrey (second-price) auction. With one item only they are identical auction formats.
- The VCG auction is **strategyproof**: it is a dominant strategy to bid one's true valuations.
- VCG may be difficult to use in practice if there are too many buyers and items: the number of assignments grows exponentially.