Fundamentals of Auction Theory

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Section 1

Introduction to auction

Auctions

Auctions are a **common** way to sell/buy goods when:

- Seller/buyer has little knowledge of what would be the "right" price.
- There is scarcity: fixed supply (Monet painting).
- The quality/quantity of the good to be sold/buy changes very frequently (electricity, fish, etc.).
- Transaction frequency is low.

What is an auction?

An auction is defined by

- **Bidding format rules**: the form of the bids
  Bids can be price only, multi-attribute, price and quantity, or quantity only.
- **Bidding process rules**:
  Closing/timing rules, available information, rules for bid improvements/counter-bids, closing conditions.
- **Price and allocation rules**:
  Final price(s) and quantities, winning bidders.

In economics an auction is commonly referred to as

- a type of **market mechanism**, which uses prices to determine allocations (who gets what).
- a **price discovery mechanism**.

Auctions are everywhere

- eBay, Art sales, house foreclosure, etc.
- Event tickets
- Treasury bonds
- Internet ads (Google, Bing, Facebook, Twitter, etc.)
- Electricity markets
- Public facilities (bridges, tunnels, etc.)
- Fish markets, Timber markets, etc.
- Daily quotation at stock exchanges
- IPO
- Course allocations at business schools
- Spectrum licences (wireless communication)

A bit of history

Historical use of auctions:

- 500 BC: evidence of women sold through an auction in Babylon
- March 28, 193 AD: Pretorian guard killed emperor Pertinax and ran an auction to choose the next emperor. Didius Julianus won (lasted 2 months only).
- Sotheby's (1744), Christie's (1766)
- Slave markets

Valuations

Each bidder in an auction has a **value** or **valuation**: the highest price she is willing to pay.

Does the valuation depends on other buyers' values?

- No $\rightarrow$ **Private value** (usually not too difficult)
- Yes $\rightarrow$ **Common value** (complicate things a lot)

In real-life it is often a mixture of the two: buyers' valuations are **interdependent**:

private value component + common value component
(common to all bidders)

In formal models of auctions bidders' valuations are often modeled this way:

- Each bidder receives a **signal** about her valuation.
- If bidders' signals are **independent** from each other and a bidder's valuation only depends on her signal then we have the **private value** model. Often (this chapter) we also assume

  that the information given by the signal is precise (i.e., there is no noise).
- If bidders' signals are **correlated** we have the **interdependent** or **common value** model.

  Usual approach for the common value model: signal is **noisy** (bidders don't get the same signal), but true valuation is the same.

  Variation (interdependent value): valuation is the sum of a private part and a common part.

hypothesis for this chapter

- We consider private values environment (independent signals)
- Each buyer knows his/her valuation (signal is not noisy).

Bidding behavior & payoffs

To what extent is my bidding related to my valuation?

- Do I bid my value? **sincere/truthful bidding**
- Do I bid something different than my value?

$$\text{payoff} = \begin{cases} \text{valuation} - \text{price paid} & \text{if win} \\ 0 & \text{if lose} \end{cases}$$

We assume that if price paid = valuation, I prefer winning the auction than not winning.

Objectives?

Take any auction (with one item to be sold),

- v is the value of the buyer (the winner of the auction).
- p is the price

$$\text{social surplus} = \underbrace{p}_{\text{seller's surplus}} + \underbrace{(v-p)}_{\text{winner's surplus}} + \underbrace{(n-1)\times 0}_{\text{losers' surplus}}$$
$$= v$$

If the winner is the **bidder with the highest valuation**, then the social surplus maximized: the auction is **efficient**.

Ideally, the bidder with the highest valuation bids the highest (but not always the case!).

Thus, efficiency $\Rightarrow$ **revenue maximization**.

Section 2

The main auctions

Ascending auctions

Ascending auction: the price starts low and goes up (gradually) until a winner is declared (or the auction is cancelled).

There are various versions of the ascending auctions, depending on **who can announce prices**. Two polar cases:

- **Only bidders announce prices**

  Such auctions are called **English outcry auctions**

  - Jump bids usually allowed.
  - The auctioneer may set a minimum increment between two bids
  - Ends when no further bid (or when reached a time limit).
  - Bidding speed matter: two bidders cannot announce **at the same time** the same bid.

  Winner: highest bidder. **Price paid** = highest announced price **by the winner**.

- **Only the auctioneer announces prices**

  Such auctions are called **English auction**

  - The auctioneer proposes a price, the bidders just say if they accept it or not. Two options:
    - Just need one bidder accepting to go to the next price.
    - All active bidders say whether they accept the proposed price. (not active $\Leftrightarrow$ refused a previous price $\Leftrightarrow$ stopped bidding).
  - Ends when there's only one bidder left (designed as the winner).

    If tie (all remaining active bidders stopped at the same price), random draw to design the winner.

  Price paid = last price announced **by the auctioneer**.

The version economists have in mind when talking about the English auction is the **Japanese button auction**:

- A price is displayed, with automatic increments (.
- Bidders push a button on a remote control.

  As soon as a bidder stops pushing the button she becomes **inactive**.
- Winner = last bidder pushing the button (if there's a tie, random draw).
  - Hence, a bidder who **stops** pushing the button **necessarily loses** the auction.

Ticking price

In an English auction the **ticking price** is the amount by which the price has to go up.

English outcry auctions may also have ticking price rules: the minimal increment for a bid (see eBay).

What is a bid?

- English auction: A **price target**

  Until which price I press the button.

  $\Rightarrow$ I bid only once.

- English outcry auction: A **price**

  $\Rightarrow$ I may bid several times.

Truthful bidding

Proposition

In the English auction, truthful bidding is a ***dominant strategy****.*
(Equivalently, the English auction is ***strategyproof****.)*

Recall that:

- A strategy describes for **any possible** bidding history (who bid, how much, when, etc.) if I bid and, if so, by how much.
- A strategy is **dominant** if it outperforms any other strategy, for any possible history.

So truthful bidding in the English auction means

"*press button until my valuation is attained (stop pressing as soon as the price is above my valuation).*"

Proof

To make it simple, suppose that my valuation is 100. Call p the displayed price of the auction, $p^* =$ winning price.

- If I keep pressing the button when $p > 100$. So, $p^* > 100$.
  - If I win, I have a negative payoff: $100 - p^* < 0$.
  - If I lose, payoff = 0

  $\Rightarrow$ my payoff is either negative or 0.

- If instead I stop pressing the button as soon as $p > 100$:
  I lose (winner is the bidder never stopping to press button).
  $\Rightarrow$ my payoff is 0.

$\Rightarrow$ stopping at $p = 100$ is better than continuing.

- If stop pressing the button when $p \leq 100$.
  - For instance I stop pressing when $p = 80$.
  - I lose the auction.
  - my payoff $= 0$.
- I choose a truthful strategy: stop pressing as soon as $p > 100$.
  - p goes above 100, I stop pressing the button, so I lose the auction, and my payoff $= 0$.
  - If for a price between 80 and 100 I am the only bidder still pressing the button:
    $\Rightarrow$ I'm the winner.
    $\Rightarrow$ My payoff $= 100 - p > 0$

$\Rightarrow$ being truthful better than stopping before $p = 100$.

Remark

In the English auction, the auctioneer learns (at the end) the valuations of all bidders except the valuation of the highest bidder.

(We will never know when she was planning to stop pressing the button.)

English $\neq$ English outcry

To see how details matter, take the (apparently similar) outcry auction.

- My valuation is 100
- Mr. Grinch doesn't like prices between 50 and 80. If the displayed price is between these two prices he bids 1,000. Otherwise, he never bids.
- If I do gradual increments in my bid, the displayed price eventually attains 50 and then jumps to 1,000.
- At 49 I should stop (momentarily) bidding or jump bid to 81: this is not (*officially*) truthful bidding.

In the end, what is an English auction?

- All the ascending auctions look similar, but they differ in their details.
- Can we define an auction that captures the key elements of an ascending auction?
  (i.e., that is not subject to little details and weird behavior like Mr. Grinch's).
- Hitherto, the ascending auction implicitly implies:
  - All bidders are participating at the same time
  - It may take some time to run the auction

  Can we define an auction that "mimics" the English auction and does not require bidders to meet and is instantaneous?

William Vickrey's idea

- Suppose there are 3 buyers with valuations $v_{\text{Pim}} = 50$, $v_{\text{Pam}} = 70$ and $v_{\text{Poum}} = 100$.
- English auction, so each buyer bid truthfully.
- Suppose the "ticking price" is 1.
- Last displayed price (= winning bid) = 70+1 = 71
- Winner is Poum

Vickrey's insight: in an English auction,

- the winner is the one with the highest valuation,
- but the price is that of the 2^{nd} highest valuation (+ ticking price).

The Vickrey auction is a **simultaneous**, **2nd price** auction (also called **2nd price sealed-bid** auction):

- Buyers submit a sealed-bid
- The winner is the one with the highest bid
- The winner pays the 2nd highest bid

  Note: the ticking price disappears because there is no dynamics.

Proposition

*In the Vickrey auction, truthful bidding is a **dominant strategy**.*

Proof

Suppose that my value is 100 and I bid 110. Let b be the highest offer by a rival.

- If $b \leq 100 < 110$:
  Then overbidding does not matter: I win and I pay b.
- If $100 < b < 110$:
  I win and I pay $b > 100$. Overbidding reduces my payoff (I would get 0 by bidding 100).
- If $b > 110$: Overbidding does not matter: I lose whether I bid 110 or 100.

Exercise: use a similar argument to show that I never want to bid less than 100.

So for both the English and the Vickrey (2nd price, sealed-bid) auctions:

- Bidders use the **same strategies**: bid their valuations
- outcome is the same (who wins & price),
- Bidders obtain the **same payoffs**,
- The seller get the **same revenue**.

Remark

- *In the English auction there is (usually) a ticking price, but there is none in the Vickrey auction.*

  $\Rightarrow$ *winner and sellers' payoffs are not* ***exactly*** *the same in both auctions.*
- *In theoretical analysis we often omit the ticking price for the English auction.*

English vs Vickrey auctions

The English and the Vickrey auctions are similar but they are not **exactly** identical:

- the Vickrey auction is a **strategic form game**: bidders can never learn the bids of their opponents.
- the English auction is an **extensive form game**: bidders *may* observe the bids of their opponents (at what price they drop out).

$\Rightarrow$ a **strategy** is a more complex object in an English auction: it specifies an action (press/stop pressing) for any possible history.

In practice the presence/absence of bidders and other hard-to-model-features can influence bidders' behavior.

$\rightarrow$ **Planet Money's podcast $\sharp$ 678: "Auction Fever"**
http://n.pr/2j4987H

If:

- bidders are rational
- bidders know perfectly their valuations (and they don't change)
- valuations are private

then bidding decisions in the English and Vickrey auctions are identical.

First-price auction

First-price auction = the winner pays her bid.

English auction: not really a first-price auction, it is more akin to a second price auction (the winner will **pay** at the 2nd highest bid).

Another possibility is to run a **first-price sealed-bid** auction:

- Bidders submit a sealed bid
- Highest bidder wins
- Winner pays her bid

In a first-price sealed-bid auction, I may not want to bid my value: If I bid my value my payoff is 0 if I win.

Let $Prob(b)$ be the probability that I win if I bid b (i.e., the probability that my bid is the highest).
If v is my valuation, my expected payoff is:

$$Prob(b) \times (v - b) + (1 - Prob(b)) \times 0 = Prob(b)(v - b)\,.$$

There is a trade-off. As I increase my bid,

- $Prob(b)$ increases (I amd more likely to win the auction);
- $(v - b)$ (my net payoff gets smaller).

Optimal strategy: choose b to maximize $Prob(b)(v - b)$.

A **strategy** of a bidder maps a valuation to a bid:

$$v \to s(v)$$

Example: If I decide to bid,

- half my valuation, then my strategy is $s(v) = \frac{v}{2}$
- valuation minus 10, then my strategy is $s(v) = v - 10$.

Assumptions

1. Bidders' strategies are symmetric: If two bidders have the same valuation, v, they bid the same, $s(v)$.
2. Bidders' strategies are **strictly increasing** in valuation.
3. For each bidder i, $s(v_i) \leq v_i$.

Assumption 1 & 2 $\Rightarrow$ winner is bidder with highest valuation .

Optimal bidding in the first-price auction

Proposition

When there are n bidders and their valuations are private, independent and are uniformly distributed in an interval $[0, X]$, and bidders' bidding strategies satisfy Assumptions 1, 2 and 3 then a bidder with valuation v bids in equilibrium

$$s(v) = \frac{n-1}{n} v$$

Proof

Assume valuations are uniformly distributed on $[0, 100]$.

- Take Alice and let v_A be her valuation.
- Assume all the other bidders bid a fraction α of their valuations

  $\Rightarrow$ A bidder with valuation v bids αv.
- $\bar{v} = \max_{i \neq \text{Alice}} v_i =$ highest valuation of all the other bidders.
- Let b be the bid of Alice. Hence

$$\text{Alice wins} \quad \Leftrightarrow \quad b > \alpha \bar{v} \quad \Leftrightarrow \quad \frac{b}{\alpha} > \bar{v} \, .$$

Without loss of generality, let Alice be the bidder $\sharp 1$.

$$
\begin{aligned}
Prob(\text{Alice wins}) &= Prob(b > \alpha \bar{v}) \\
&= Prob(\bar{v} < \frac{b}{\alpha}) \\
&= Prob(\text{valuations of all bidders except Alice } < \frac{b}{\alpha}) \\
&= Prob(v_2 < \frac{b}{\alpha}) \times Prob(v_3 < \frac{b}{\alpha}) \times \dots \\
&\quad \cdots \times Prob(v_n < \frac{b}{\alpha}) \\
&= \underbrace{\frac{b}{100\alpha} \times \frac{b}{100\alpha} \times \cdots \times \frac{b}{100\alpha}}_{n-1 \text{ times}} = \left(\frac{b}{100\alpha}\right)^{n-1}
\end{aligned}
$$

So Alice's expected payoff when bidding b is

$$E(u_{\text{Alice}}(b)) = \left(\frac{b}{100\alpha}\right)^{n-1} (v_{\text{Alice}} - b) \ .$$

Optimal bid found taking the derivative equal to 0.

$$\begin{aligned}
& \frac{\partial E(u_{\text{Alice}}(b))}{\partial b} = 0 \\
\Leftrightarrow \quad & \frac{1}{100\alpha}\left(\frac{b}{100\alpha}\right)^{n-2}\Bigg((n-1)(v_{\text{Alice}} - b) - b\Bigg) = 0 \\
\Leftrightarrow \quad & (n-1) \times (v_i - b) = b \\
\Leftrightarrow \quad & b = \frac{n-1}{n} v_{\text{Alice}} \ .
\end{aligned}$$

Optimal bid does not depend on α! So we can set $\alpha = \frac{n}{n-1}$.

For each bidder, if all the other bidders bid α their valuation then it is optimal to also bid $\alpha \times v_{\text{Alice}}$.

We have a symmetric equilibrium.

More generally:

$$s(v_i) = E(\max_{j \neq i} v_j \mid v_j \leq v_i \text{ for all } j \neq i),$$

This is the **expected second highest valuation conditional on v_i being the highest valuation**.

Dutch auction

The **Dutch/descending price** auction is a first-price auction:

- There is a **price clock**: displays a price that is decreasing
- The auction stops as soon as someone says “Mine!”
- **Strategically equivalent** to a first-price sealed-bid auction: bidding one’s valuation is **not** a dominant strategy (in practice, not so equivalent).

  Strategic equivalence because both first-price and Dutch auctions are strategic form games with identical strategy sets (bids) and payoff functions.

Advantages:

- Can go (very) fast
- Can use bidders' nervousness (so that they bid higher than their value).

Examples of Dutch auctions:

- https://youtu.be/JKn5_8rrRoo
- https://youtu.be/gq0wsXOKlL4

Dutch vs first-price auction

Dutch and first price look different:

- First-price auction is a **strategic form game**
- Dutch auction is an **extensive form game**.

But in the Dutch auction a bidder can never distinguish between any two histories unless another bidder said "Mine!"
But in this case it is too late, the bidder has no action to take.
$\Rightarrow$ Strategies are identical in both auctions: a price.

- Bidders use the **same strategies**,
- Bidders obtain the **same payoffs**,
- The seller get the **same revenue**.

$\Rightarrow$ Dutch and first-price sealed-bid auctions are **equivalent**.

Which auction format?

What we've seen so far:

- English auction ⇔ second-price sealed-bid auction
- Dutch auction ⇔ first-price sealed-bid auction

But the English and Dutch auctions are **not** the same! bidders' strategies differ.

Yet...

Let us compare the sellers' expected revenue between the first-price and second-price auctions.

- n bidders, valuations random between 0 and 100 (i.e., uniform distribution).
- Second-price auction:
  - bidders bid their valuations.
  - Winner pays 2nd highest bid

$$\begin{aligned} & E(\text{Seller's revenue}) \\ = & E(\text{second highest valuation}) \\ = & 100\frac{n-1}{n+1} \end{aligned}$$

- First-price auction:
  - bidders bid less than their valuations.
  - Winner pays her bid

$$
\begin{aligned}
& E(\text{Seller's revenue}) \\
&= E(\text{highest bid}) \\
&= E\left(\frac{n-1}{n}\text{highest valuation}\right) \\
&= \frac{n-1}{n}E\,(\text{highest valuation}) \\
&= \frac{n-1}{n} \times 100 \times \frac{n}{n+1} = 100\frac{n-1}{n+1}.
\end{aligned}
$$

- Seller's expected revenue is the same for first and 2nd-price auctions!

Revenue Equivalence

Theorem

Assume that there are n bidders, and their valuations are private and identically and independently distributed (from a distribution over an interval $[\underline{v}, \bar{v}]$ that has a strictly increasing cumulative probability distribution).
Then ***any auction*** *such that*

- *(i) the winner is always the bidder with the highest valuation; and*
- *(ii) the bidder with the lowest valuation, $\underline{v}$ has a zero expected payoff,*

yields in equilibrium the ***same expected revenue*** *for the seller and the expected price made by any buyer is the same.*

Proof

Consider **any** auction such that:

- The winner (in equilibrium) is the bidder with the highest valuation

For any bidder i, let $U_i(v_i)$ be the **expected payoff in equilibrium**.

$$U_i(v_i) = \underbrace{v_i Prob(v_i)}_{\text{Doesn't depend on auction}} - \underbrace{E(\text{payment if wins when valuation } = v_i)}_{\text{Depends on auction}}$$

Suppose bidder (with valuation v_i) uses the bid that is optimal for another valuation, $\widetilde{v}$, and get expected payoff $U(\widetilde{v})$.

Since we have an equilibrium, we have

$$U_i(v_i) \geq U(\widetilde{v})$$

where

$$\begin{aligned} U(\widetilde{v}) &= v_i Prob(\widetilde{v}) - E(\text{payment if wins when valuation } = \widetilde{v}) \\ &= v_i Prob(\widetilde{v}) - E(\text{payment if wins when valuation } = \widetilde{v}) \\ &\quad + \widetilde{v} Prob(\widetilde{v}) - \widetilde{v} Prob(\widetilde{v}) \\ &= U_i(\widetilde{v}) + (v_i - \widetilde{v}) Prob(\widetilde{v}) \end{aligned}$$

But $U_i(v_i) \geq U(\widetilde{v})$.

$$\Rightarrow \qquad U_i(v_i) \geq U_i(\widetilde{v}) + (v_i - \widetilde{v}) Prob(\widetilde{v})$$

Let $\widetilde{v} = v_i + dv$

$$\Rightarrow \qquad U_i(v_i) \geq U_i(v_i + dv) - dvProb(v_i + dv) \tag{1}$$

Take now a bidder whose valuation is $v_i + dv$ and do the same (reporting v_i instead of $v_i + dv$):

$$U_i(v_i + dv) \geq U_i(v_i) + dvProb(v_i) \tag{2}$$

(1) & (2) together give

$$Prob(v_i + dv) \geq \frac{U_i(v_i + dv) - U_i(v_i)}{dv} \geq Prob(v_i)$$

and thus

$$\frac{dU_i}{dv} = Prob(v)$$

So we have, with $\underline{v}$ being the lowest valuation,

$$U_i(v_i) = U_i(\underline{v}) + \int_{x=\underline{v}}^{v_i} Prob(x)dx \ . \tag{3}$$

Take two different auctions, but

- without changing the bidders and the payoff of the bidder with the lowest valuation
- and assume that this bidder loses the auction with probability 1

Then

- Expected payoff given by (3)
- It does not depend on the auction format!

- Take any two auctions (that satisfy the hypothesis of the theorem)
- Eq. (3) says that bidders' expected payoffs does not depend on the auction.
- $\Rightarrow$ expected payment does not depend on the auction.
- $\Rightarrow$ sellers' expected revenue does not depend on the auction.

Reserve price

In 1990 auctions for radio spectrum in New Zealand:

- Highest bid = NZ$ 100,000, 2nd highest bid = NZ$ 6.
- Highest bid = NZ$ 7 Millions, 2nd highest bid = NZ$ 5,000.

Solution:
Add a **reserve price**.

- Only bids above the reserve price are considered.
- Guarantee a minimum revenue for the seller
- A bit equivalent to the seller placing a bid!
- But risky: if no bid meet the reserve price final revenue is 0.

Reserve price with a second-price auction

- $r =$ be the reserve price
- $b_1 =$ highest bid
- $b_2 =$ second highest bid

$$\text{Seller's revenue} = \begin{cases} 0 & \text{if } b_1 < r \\ r & \text{if } b_2 \leq r \leq b_1 \\ b_2 & \text{if } r < b_2 \end{cases}$$

Optimal reserve price

- We want the reserve price that maximizes the revenue
- Ideally, set $r =$ highest valuation.
- Trade off: A higher reserve price gives
  - higher revenue (if sell)
  - lower probability to sell.

One bidder case.

- Valuation v, unknown. Assume that v is random between 0 and 100.
- With reserve price r,

$$\text{seller's revenue} = \begin{cases} r & \text{if sell} \\ 0 & \text{if don't sell} \end{cases}$$

So we have

$$\begin{aligned} E(Revenue) &= Prob(\text{sell}) \times r + Prob(\text{not sell}) \times 0 \\ &= Prob(v \geq r) \times r \\ &= (1 - Prob(v \leq r)) \times r \\ &= r \times \left(1 - \frac{r}{100}\right) \end{aligned}$$

$\Rightarrow$ Expected revenue maximized for $r = 50$.
In fact, the optimal reserve price does **not** depend on the number of bidders!

Optimal auction

Theorem (Myerson (1981), Riley & Samuelson (1981))

Let F a the cumulative distribution (and f the corresponding density function). If bidders's valuations are drawn independently from the distribution F then a seller's optimal auction consists of choosing and running a Vickrey auction with the reserve price r that satisfies $\psi(r) = 0$, where

$$\psi(r) = r - \frac{1 - F(r)}{f(r)}. \tag{4}$$

$\psi(v)$ is the **virtual valuation** of a bidder with value v

Virtual valuation

- The virtual valuation of a bidder is the marginal revenue brought by the bidder.
- A bidder with valuation v "would buy" a quantity of $1 - F(v)$. (buying with probability 30% $\approx$ buying 30% of the good).
- So $q(p) = 1 - F(p)$ is the demand function
- So $p(q) = F^{-1}(1 - q)$ is the inverse demand function
- So revenue of the seller is

$$R(q) = p(q) \times q = qF^{-1}(1 - q).$$

- But seller has zero cost! (Good is already produced).

- Let's maximize the revenue

$$\text{Marginal revenue} = R'(q) = F^{-1}(1-q) - \frac{q}{F'\left(F^{-1}(1-q)\right)}.$$

Which yields

$$MR(p) = p - \frac{1-F(p)}{f(p)} = \psi(p).$$

Hence, the virtual valuation of a buyer, $\psi(p)$, captures the marginal revenue brought by that buyer.

Example

- Bidders' valuations are uniformly distributed on $[0, 100]$.
- cumulative distribution is

$$F(v) = \frac{v}{100}$$

and thus the density function is

$$f(v) = \frac{1}{100}$$

- So virtual valuation is

$$\psi(v) = v - \frac{1 - \frac{v}{100}}{\frac{1}{100}} = 2v - 100.$$

- Recall that profit is maximized when r such that $\psi(r) = 0$. The solution is $r = 50$.

But calculating the "perfect" reserve price can be tricky...

Theorem (Bulow & Klemperer, 1996)

When valuations are private and independent, a second-price auction with $n+1$ *bidders give a higher expected revenue than an Optimal Mechanism with* n *bidders.*

1 bidder (with optimal reserve) vs 2 bidders (no reserve)
Uniform distribution on $[0, 100]$.

- 1 bidder with optimal reserve:
  - optimal reserve = 50
  - Expected revenue:
$$\frac{1}{2} \times 50 = 25$$
- 2 bidders without reserve:
  - Expected revenue:
$$100\frac{n-1}{n+1} = \frac{100}{3} = 33.333\ldots$$

Proof of Bulow–Klemperer's result: Kirkegaard's argument

Definition

An auction is **constrained optimal** if among all auctions such that object sold with **certainty** it maximizes the revenue.

Remark: Vickrey auction with optimal reserve is **not** constrained optimal.

- if $r <$ highest valuation then no bidder buys the good
  $\Rightarrow$ Object not sold with **certainty**.

Kirkegaard auction with $(n+1)$ bidders:

Step 1 Run Vickrey auction with n bidders and optimal reserve.
If object not sold go to step 2.

Step 2 Sell object for \$0 to $(n+1)$-th bidder.

- With the Kirkegaard auction object sold with certainty.
- Revenue at least as good as the optimal auction with reserve (and n bidders:

$$\begin{aligned} &\text{Expected revenue Kirkegaard } (n+1)\text{-auction} \\ =\ &\text{Expected revenue Optimal } n\text{-auction.} \end{aligned}$$

- The English auction is constrained optimal.
- So

$$\begin{aligned} &\text{Expected revenue Kirkegaard } (n+1)\text{-auction} \\ &< \text{Expected revenue English } (n+1)\text{-bidders.} \end{aligned}$$

and thus

$$\begin{aligned} &= \text{Expected revenue Optimal } n\text{-auction.} \\ &< \text{Expected revenue English } (n+1)\text{-bidders.} \end{aligned}$$

About the Revenue Equivalence Theorem

Source of **two** Nobel Prizes:

- William Vickrey in 1996 for the first version.
- Roger Myerson in 2007 for the use of a generalization to characterize the optimal mechanisms.

Take-away

- The English and the Dutch auctions are **dynamic** auctions: there is a ticking price.
- The Vickrey (second-price) and the first-price auctions are **simultaneous** auctions (there is no ticking price).
- Truthful bidding is a dominant strategy for both the English and Vickrey auctions. They yield the same outcome.
- Truthful bidding is **not** a dominant strategy for the Dutch and first-price auctions (and they are strategically equivalent).

- The seller's expected revenue is the same for the English, Dutch, Vickrey and first-price auctions.
- An optimal auction is an auction that maximizes the seller's revenue. It can be done using a Vickrey auction with optimal reserve prices (Myerson (1981), Riley & Samuelson (1981)).
- Additional bidders and run an English auction can yield higher revenue than in an optimal auction (Bulow & Klemperer (1996)).